A GENERAL THEORY FOR THE MENSURATION OF THE ANGLE SUBTENDED BY TWO OBJECTS, OF WHICH ONE IS OBSERVED BY RAYS AFTER TWO REFLECTIONS FROM PLANE SURFACES, AND THE OTHER BY RAYS COMING DIRECTLY TO THE SPECTATOR'S EYE. BY GEORGE ATWOOD, M. A. F. R. S.

THE actual determination of an angle implies two obſervations, one taken at each extremity of the arc by which that angle is meaſured. When fixed aſtronomical quadrants or other ſectors are uſed for the practical eſtimation of angles, one of theſe obſervations is previouſly made by directing the axis of the teleſcope or line of collimation to ſome fixed point in the heavens, the index being then coincident with the initial point on the arc of the ſector: after this adjuſtment one obſervation only is neceſſary to aſcertain the angular

diftance between that point and any other celeftial object in the plane of the fector. This method, however, is evidently impracticable, unlefs the inftrument can be fteadily fixed; for which reafon aftronomical quadrants become ufelefs at fea; and from the difficulties which attend placing them in their due pofition and adjuftment on firm ground, they are almoft wholly confined to regular obfervatories.

Mr. HADLEY*, by an ingenious application of optical principles, contrived to bring both extremities of the arc meafured into the field of the fpectators's view at the fame time; by which improvement, angles are taken at fea, as well as on land with an unfixed inftrument, to a degree of accuracy fufficient for nautical and other purpofes, when the utmoft exactnefs is not required.

Mr. HADLEY'S invention is a particular cafe of a very extenfive theory, as yet but little attended to. According to his method, which is well known, the two reflecting furfaces ufed in the obfervation are perpendicular to the plane of motion; the direction of the telefcope, and of the rays paffing between the reflectors being parallel to that plane; whereas the inclination of the telefcope, and of the intermediate rays, as well as of the reflectors themfelves to the plane of motion, admits of unlimited variety. A general theory to determine the angle obferved by two reflections from the data on which its magnitude depends, without limitation or reftriction, feems applicable to feveral ufeful purpofes in practical aftronomy. Having never feen any geometrical conftruction or analyfis of this curious problem, I was induced to beftow fome confideration on

* Phil. Tranf. N° 420. See alfo a tract, intituled, The Theory of HADLEY'S quadrant, by the rev. W. LUDLAM.

on the ſubject, and ſhall be happy if the reſult of my inquiries appears to merit the attention of the Royal Society.

Art. 1. The manner of taking an obſervation by two reflections unconfined to any particular caſe may be deſcribed thus. Let C, B (fig. 1.) repreſent two plane reflecting ſurfaces, inclined to a plane OPA at any given angle. Through any point of the reflecting ſurface C draw a line perpendicular to the plane OPA, and with the point where the line meets the plane as a centre (which muſt here be repreſented by C) and any diſtance CP, deſcribe a circle OPA. The reflecting plane B always continuing fixed, let the reflector C be moveable along with the radius CP as it revolves in the plane OPA round the centre C: the angular motion of the ſpeculum C, referred to the circumference OPA, will be meaſured by the arc which the radius CP deſcribes, the inclination of the plane C to the plane of motion OPA being always the ſame, and equal to that of the fixed ſpeculum B.

2. The two plane reflectors, B and C, being equally inclined to the plane OPA, it follows, that during the motion of C there muſt be ſome point O in the circumference OAP, at which when CP arrives, the reflector C will be parallel to the fixed reflector B.

3. When the moveable radius which carries round the plane C is at any other poſition CP, let a ray flowing from a diſtant object T impinge on the ſpeculum C; let it be reflected from thence in the direction CB, and being again reflected at B in the direction BG, let it be obſerved by a ſpectator's eye at G; the image of T will appear ſomewhere in the line GBS; ſuppoſe that a ray flows from a diſtant object S ſituated in the line GB produced, and that this ray SG comes directly to the ſpectator's

eye at G: the object S feen by direct rays, and the image of the point T feen by rays after two reflections, will appear to coincide in the line GBS. This is an obfervation by two reflections, from which, together with fuch data as limit the problem, the true angle fubtended by the objects T and S is to be inferred.

4. The data which limit this problem, being neceffary for the determination of the angle fubtended by T and S are in number four, which are next to be confidered. 1ft. One of thefe data is the arc PO, being the angular diftance of the moveable radius CP, meafured on the circumference of the circle OPA, from that pofition CO, at which the two reflectors are parallel; the fituation of this arc OP in refpect of the point O being fuppofed known, that is, it being known on which fide of that point, OP is fituated in refpect of the ray BG: 2dly, The common inclination of the reflecting planes B and C to the plane of motion is another of thefe data. The third and fourth of the conditions muft be mentioned rather more particularly. The ray BG is always underftood to be given in pofition in refpect of the plane of motion OPA (confidered as immoveable) being either coincident with the line of collimation of a telefcope, or directed by fights fo as to be invariably fixed: the fpeculum B alfo being unmoved, the line or ray BC will never change its pofition, from the known principles of reflection. The angle CBG, therefore, and the half of that angle being the angle of incidence at which CB impinges on B, will be always of the fame magnitude; whereas the half of the angle BCT, or the angle of incidence on the moveable fpeculum C, is continually changing, while C is carried round in the plane of motion: this conftant angle of incidence or reflection at the fixed fpeculum B will be another of the data neceffary to determine the problem.

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The rays GB, BC, and the speculum B, being fixed in respect of each other, and of the plane OAP, the plane CBG will also be given in position; that is, its inclination to the plane of motion, or to any other fixed plane, will constantly be the same: whereas the inclination of the plane BCT to the plane of motion, or other fixed plane, will be continually changing while the reflector C revolves with the radius CP. The position of the plane GBC constitutes the fourth and last of the data, and it will be immaterial to what fixed plane it is referred. In the ensuing solution the situation of this plane will be defined by its inclination to the fixed secondary of the plane of motion which passes through the point O.

5. The enumeration of these data leads to the construction of the problem, a few observations being previously inserted to prevent repetitions and unnecessary references. 1st, The objects observed are understood to be lucid or illumined points, and so distant, that the rays which flow from either of them may be esteemed parallel without error as far regards these observations: such objects are the fixed stars, any given points in the disks of the sun or planets, &c. 2dly, As in measuring the angular positions of objects which lie in the same plane, these objects are referred to the circumference of a circle, the centre of which is coincident with the spectator's eye; so in estimating the positions of objects which lie in different planes, and of the inclinations of these planes to each other, the objects, &c. are referred to the circumference of a sphere, of which the centre coincides with the centre of the spectator's view: applying this to the present case, since the lines CT, CB, SG (fig. 1.) are situated in different planes; in order to estimate their positions, any point may be assumed as the centre of a sphere, and through that point lines are to be drawn parallel to the given lines CT, CB, SG, the points in

which the lines interfect the fpheres furface will give their relative fituations by the rules of trigonomety. 3dly, There will be no neceffity to reprefent the reflecting planes in the general conftruction, fince the pofitions of the perpendiculars to the planes will give the fituations of the planes themfelves.

6. To determine by conftruction the angle fubtended by the objects T, S, from the data which have been defcribed, let APOCQ (fig. 2.) reprefent a great circle of the fphere to the furface of which the objects obferved, and the pofitions of the incident and reflected rays, &c. are referred; C being the center, CK the axis, and K the pole of this great circle; through K draw any fecondary KO, and from the pole K, at the diftance of the arc KF, = the meafure of the given inclination of the reflecting planes to the plane of motion, defcribe a parallel or leffer circle FIM: with the pole F, and at a diftance equal to a quadrant, defcribe an arc of a great circle interfecting the fecondary KO produced in the point X, and in this arc from X take XY = the meafure of the given inclination of the fixed plane of reflection at the fpeculum B to the fecondary which paffes through the point O; and draw the quadrant YF, which produce in the direction YF: from F on either fide of F fet off FD equal to the meafure of the given conftant angle of incidence at the fpeculum B, and make FB (taken on that fide of F which is oppofite to D) equal to FD. Draw the radius CO: from O fet off an arc OP in the circumference OPA equal to the meafure of the angular diftance defcribed by the moveable radius CP from that pofition at which the reflectors are parallel; obferving that the arc OP be on that fide of the point O which* correfponds with the conditions of the problem (art. 4.): through P defcribe the fecondary KP interfecting the parallel FIM in the point I: through B and I defcribe

* It is fuppofed to be known, whether CP beginning its motion from the pofition CO approaches towards the vifual ray BG or recedes from it.

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the arc of a great circle BIE, and in it take EI equal to IB: through D and E draw the arc of a great circle DE: the arc DE will be the measure of the true angle subtended by the objects observed, according to the data of the problem.

Previous to the demonstration of this construction, the application of it to the method of observation by two reflections should be described. Join CP, CI, and CF. To the extremity C of the radius CP let a plane speculum be affixed, CI being always perpendicular to this plane: as PC revolves in the plane of motion, the perpendicular CI will describe the parallel or lesser circle FIM, and when CP coincides with CO, CI will coincide with CF. Through B draw BR parallel to CF, and let a plane speculum be fixed at B perpendicular to BR; CF and BR being parallel when the perpendicular CI coincides with CF, the reflectors at C and B will then be parallel.

Join CD, and produce it to a very distant point S, and through B draw GBS parallel to CDS; the reflectors C and B being parallel, and their perpendiculars coinciding with CF and BR, let a ray SC impinge on the reflector C: because FC is the perpendicular to the speculum C and the arc DF = FB by construction, these arcs being in the plane of the same great circle DBQ, it follows, that the ray SC will be reflected from C in the direction CB, impinging on the speculum B at the angle of incidence CBR; and since DC and BG are parallel by construction, and the parallel lines FC BR fall on them, the angles RBG, FCD, will be equal, and FCB or CBR = RBG. CB therefore being the ray incident on the speculum B will be reflected in the direction BG parallel to SC; and a ray SG coming directly from S will be seen coincident with the reflected ray BG. Here we observe, that the planes of reflection at C and B, that is, the planes DCB and CBG coincide, the reflectors being parallel.

Let the radius CP move from the poſition CO, carrying with it the ſpeculum C and its perpendicular CI: then, EI being equal to IB by conſtruction, a ray impinging on C in the direction TEC will be reflected in the plane ECB, and becauſe ECI = ICB, the reflected ray will coincide with the line CB, and after reflection at B will * proceed in the direction BG, being coincident with the ray SG which comes directly from S. When the perpendicular CI leaves CF, the plane of reflection ICB becomes inclined to the plane of reflection DCBG with which it before coincided; but the poſition of the rays CB, BG, and of the perpendicular BR, remains unaltered; for which reaſon the plane GBCFD correſponds to the fixed plane of reflection deſcribed among the conditions (art. 4.). When CI was coincident with CF, the radius CP was coincident with CO, O being the initial point of the arc OP, deſcribed by the radius CP, denoting that when CP coincides with O, the reflectors being then parallel, the inclination of the ray SC obſerved after two reflections, and SG obſerved by direct rays parallel to SC, is nothing: the great circle KO, therefore, which paſſes through O and F, will be the fixed or primitive ſecondary to which the inclination of the fixed plane of reflection at the ſpeculum B is referred.

The demonſtration of the conſtruction will conſiſt of two parts. It muſt be firſt ſhewn, that the conditions or data of the problem are obſerved in the conſtruction. 2dly, That the magnitude of the arc ED, which meaſures the angle ſubtended by the obſerved objects is limited or determined by them.

Suppoſing the angle TCS to be of any unknown quantity, it has appeared, that according to the conſtruction, the rays which come from T, and are ſeen after two reflections at C and B, will be obſerved to coincide with the rays which come

* Supra.

directly

directly from S. That the conditions of the problem are fulfilled in the construction is demonstrated thus:

1st, The inclination of the reflectors B and C to the plane of motion was constructed of the magnitude which is measured by the arc KF. KC is perpendicular to the plane of motion. CF is perpendicular to the reflector C, and the inclination of these two lines CK, CF, is measured by the arc KF; but the inclination of any two planes is the same as the inclination of two lines which are perpendicular to them; the inclination therefore of the reflector C to the plane of motion is measured by the arc KF, and the speculum B is equally inclined to the plane of motion with C by the construction, the perpendiculars CF and CI being parallel when both are situated in the plane of the same great circle DBQ.

2dly, KO being the secondary to which the position of the fixed plane of reflection DFB at the speculum B was referred, that given inclination will be equal to the angle OFB, which is measured by the arc XY according to the construction, FY being a quadrant.

3dly, Moreover, FD = FB, was constructed equal to the constant angle of incidence at the fixed speculum; CBR is the angle of incidence at the fixed speculum B, and it is equal to the angle BCF, because CF and BR are parallel by construction, and CB falls on them; FB. or its equal FD therefore is truly constructed the measure of the given constant angle of incidence at the fixed speculum B.

4thly, Because it has been shewn that CO is the position of the radius CP, when the reflectors are parallel, the arc OP is rightly constructed the measure of the angular distance of the radius CP from that position.

It remains only to demonstrate that these four given quantities, KF, OP, XY, and DF, limit the magnitude of the arc

ED:

ED: through I and F draw the arc IF: then the given arc KF or KI, and the angle IKF, meafured by the given arc PO, define the triangle IKF, and in it, therefore, the fide IF and and the angle IFK are determined. If from IFK, the given angle DFK, meafured by the arc XY, be fubtracted, the remainder IFD, and IFB its fupplement to 180° will be defined: the given arc FB, with the angle IFB and the arc IF, determine the angle IBF, and the arc IB, or its double BE: and the given arc BD, the arc BE, with the contained angle DBE before determined, define the arc ED, which is therefore the true meafure of the angle fubtended by the objects obferved under the conditions fulfilled in the conftruction.

7. The computation of the obferved angle DCE being for the prefent omitted, fome confequences which follow from the conftruction may be inferted in this place, being either corollaries, or fuch truths as admit of eafy geometrical deduction from the general propofition. The line DC will always be the pofition of the vifual ray or line of collimation of the telefcope ufed in the obfervation, and the inclination of it to the plane of motion will be meafured by the complement of the arc DK to a quadrant. The line BC will be the pofition of the ray which paffes between the reflectors B and C, and the inclination of it to the plane of motion will be meafured by the complement of the arc BK to a quadrant. Thefe arcs are left out of the figure, that the more material parts of the conftruction might not be confufed by them.

8. Every thing elfe remaining, let the parallel FIM (fig. 3.) be projected on the plane of motion QOP. Through the points F and I draw the arc of a great circle NIFR. The obferved * objects T and S, or, which is the fame thing, the points of interfection at the fphere's furface E and D will be at equal

* Compare fig. 2.

perpen-

perpendicular distances from this arc, which may be demonstrated thus. Through the points E, D, and B, draw the arcs EN, DL, and BR, perpendicular to NIFR: then the triangles DFL, FBR, being equal, DL will be equal to BR; moreover, the triangles ENI, IRB, being equal, the arcs EN, RB, will be equal: from whence it follows, that $EN = DL$, or the perpendicular distances of the points E and D from the arc of a great circle which passes through the points I and F, are equal. It appears also, from the same construction, that the arc NL, intercepted between the two perpendiculars EN, DL, is equal to twice IF: for because the triangles EIN, RIB, are equal, as are the triangles DLF, RFB, it follows, that NI is equal to IR, and LF to FR, wherefore $2IR = NR$, and $2RF = LR$: whence, by subtracting equals from equals, $2RI - 2RF = NR - LR$, or $2IF = NL$, which was the equality to be demonstrated.

9. From this last construction and demonstration the following proportion is inferred. As radius : cosine of DL or EN, so is the sine of IF to the sine of half the arc ED, or of half the observed angle: for if the arcs NE, LD (fig. 3.), be continued until they meet in the pole H, the arcs NH, LH, will be quadrants, and the triangle EHD isosceles, which, from a property of spherics too obvious to need demonstrating, gives this proportion: as the chord of NL to the chord of ED, so is radius to the sine of DH, or cosine of DL; but the chord of NL is equal to the chord of 2IF from art. 8. We have, therefore, as radius : cosine DL, so is the chord of 2FI to the chord of ED, or, which is the same proportion, as radius : cosine DL, so is the sine of IF to the sine of half ED.

10. From the last article it appears, that the sine of half the angle between the observed objects, or the sine of half ED,

is proportional to the fine of FI and the cofine of DL jointly; confequently the fine of FI being the fame, fin. $\frac{1}{2}$ ED is proportional to the cofine of DL; this will lead to the reafon why in enumerating (art. 4.) the * conditions which limit the magnitude of the obferved arc ED, the pofition of the fecondary KP, in refpect of the point of interfection Q and of the fixed fecondary KO, was annexed: for it will appear, that every thing elfe being the fame, the magnitude of the arc ED will depend on the pofition of the fecondary KP, whether it be on one fide of the fixed fecondary KO, or on the other, the angles PKO, pKO, being equal. Having fet off Op = OP draw the fecondary Kp interfecting the parallel FIMU in the point U; and through B and U draw the arc of a great circle BUW; take UW = BU; and through D and W draw the arc of a great circle DW: then by the conftruction and demonftration in art. 6. the angle fubtended by the obferved objects will be meafured by the arc DW, and it will be eafy to fhew, that DW is not equal to DE, except in two extreme cafes; that is, when the fixed plane of reflection DFB is either coincident with the primitive fecondary KO or perpendicular to it. Through the points F and U draw the arc of a great circle VFU, and from D draw the arc DV perpendicular to VFU: fince † the fines of half the arcs DE, DW, are in a proportion compounded of the proportions of the fine of IF to the fine of FU, and of the cofine of DL to the cofine of DV, the fines of IF, FU, being equal by the conftruction, the fines of half the arcs ED, DW, will be in the fame proportion with the cofines of DL and DV, which are evidently unequal; confequently, the fines of half the arcs DE, DW, and therefore the arcs themfelves, muft be unequal.

11. The angles PKO, OKp, remaining equal, when the fixed plane of reflection BFD (fig. 4.) is coincident with the

* Compare fig. 2. † Supra.

fecondary

secondary KO, or at right angles to it, the perpendiculars DL, DV, become equal in both cases, which is obvious from the equality of the triangles DVF, DLF; it follows, therefore, (art. 10.) that the sines of FI and FU, and the cosines of DL, DV, being equal, the arcs DE, DW, will be equal in these two extreme cases, but in no other.

12. Since the angle subtended by the observed objects (art. 10.) depends only on the sine of IF and the cosine of DL, it is plain, that if the points D and B be interchanged, (fig. 2, 3, 4.) the angle observed will not be altered, every thing else remaining the same; because neither the sine of IF, nor the cosine of DL, is affected by this change. For this reason in any construction for measuring angles by two reflections, the position of the * visual ray may be altered into that of the ray BC passing between the reflectors, which will become in that case the situation of the visual ray, this alteration noways affecting the observed angles.

13. While the perpendicular CI (fig. 2. and 5.) describes the parallel FIM, the angle of incidence on the moveable speculum C, that is, the angle ECI or ICB, measured by the arc BI, continually increases until it arrives at a certain limit. This limit is determined by drawing through the points B and K the arc of a great circle BKM. When the perpendicular CI arrives at M, the arc BM is the greatest possible, which will therefore be the measure of the greatest angle of incidence on the moveable speculum, according to this construction, the radius CP having then described from O an arc which is the measure of the angle FKM. Now it is plain, that if the arc MB should be greater than a quadrant, there can be no vision by two reflections, when the perpendicular CI coincides with M (supposing the moveable speculum to reflect on one side only) because the angles of incidence and reflection on any speculum

* The position of the ray DC is the same with that of the ray BG parallel to it, when referred to distant objects.

must

muft be lefs than 90°. If BM be lefs than a quadrant, an obfervation by two reflections may be taken when the radius CP is directed to any point in the circumference of the plane of motion. When the arc BM is greater than a quadrant, two other limits will be produced in the circumference of the plane OCP; while the radius CP is between thefe limits, no obfervation by two reflections can be taken: thefe limits are conftructed thus (fig. 6.). BM being greater than a quadrant, with the pole B and diftance BI equal to a quadrant, defcribe the arc of a great circle I*i* interfecting the parallel FIM in the points I and *i*: through I and *i* draw the fecondaries KY, KZ: while the radius CP is between Z, and Y no obfervation can be taken by two reflections. If BIE, B*i*E, be drawn equal to a femi-circle, and DE joined; then DE will be the meafure of the limiting angle which can be obferved by this conftruction, either on one fide of KO or on the other; and becaufe, by the principles of trigonometry, the arcs BD and DE are in the fame great circle, BDE being a femi-circle, we fhall derive from the conftruction this conclufion: the difference between 180° and double the angle of incidence on the fixed fpeculum, will be a limit which terminates the angle obferved by two reflections in every cafe, when the arc BM is greater than a quadrant *.

14. In any given example formed on the principles which have been demonftrated (fig. 2.) for the eftimation of angles by two reflections, three of the four quantities neceffary to determine the refult muft conftantly be the fame, while the fourth, that is, the arc OP, varies with the magnitude of the angle fubtended by the objects obferved: the different magnitudes of thefe three given quantities will caufe a great variety of properties in conftructions which depend on the general

* This termination of the angle which can be obferved by two reflections may happen while the obferved angle is increafing or decreafing during the revolution of the index in the plane of motion.

theory.

theory. If the angle DFK (fig. 2.), being the inclination of the fixed plane of reflection to the primitive seconda y be $= 90°$, and the arc KF, or the inclination of the reflectors to the plane of motion, be $= 90°$ also, the construction will become that of HADLEY's instrument (fig. 7.), whatever be the magnitude of the arc DF, that is, of the angle of incidence on the fixed speculum B: in this case the points F and O, and the points I and P, coincide. Here IF or PO measures the inclination of the reflectors to each other; and because BF = FD, and BI = IE, by construction, it follows, that DE = 2PO, that is, the angle subtended by the observed objects is double to the angle at which the reflectors are inclined to each other. This is a known property of HADLEY's instrument, in which the visual ray, and the ray intermediate between the reflectors, are in the plane of motion, which is also expressed in the construction, DC and BC coinciding with the plane POC.

15. Bisect KO in F; then will KF = 45° (fig. 8.). The visual ray CD being coincident with the plane of motion, let the inclination of the reflectors to that plane be equal to 45°: moreover, let the angle * DFK = 180°; so shall D coincide with O, and B with K: this will afford a good example to the general theory. Let the radius CP move into any given position, carrying with it the speculum C and its perpendicular CI: here the observed object E and the point B are always equi-distant from I; and because BI is half a quadrant by construction, it follows, that IE will be of the same magnitude, BE therefore will be a quadrant, and consequently E will coincide with P, being always in the plane of motion. The following properties are also derived from this construction. 1st, The arc DE subtended by the observed objects is equal to the arc described by the index or moveable radius CP from O;

* Compare fig. 7.

differing

differing in this from Mr. HADLEY's conſtruction, in which the angle obſerved is equal to double the angle deſcribed by the moveable radius from the initial point of the arc O. While therefore the moveable ſpeculum C is carried round by the radius CP in the plane of motion according to the new conſtruction juſt deſcribed, the image of E moves with an angular velocity juſt equal to that of the radius, the motion of the image being, according to Mr. HADLEY's invention, always greater than that of the radius in the proportion of 2 to 1. 2dly, The angles of incidence and reflection on both ſurfaces are conſtantly the ſame, being equal to 45°. 3dly, BI (art. 13) being always leſs than 90°, obſervations by two reflections may be taken all round the circle, that is, angles of any magnitude may be meaſured by this conſtruction. It will not be difficult in practice to regulate the inclination of the plane reflectors to the plane of motion, with the other given quantities to their true magnitude. Let the reflectors B and C be brought parallel when the index or radius CP is directed to O, being the initial point of the arc OP: in order to examine whether the fixed plane of reflection BFD be coincident with the primitive ſecondary KO, it is only neceſſary to obſerve the angle ſubtended by two given objects when the index CP is on the different ſides of the initial point O: if the index be directed to unequal diſtances from that point at the times of obſervation, a correction is required (art. 11.). To examine whether the inclination of the reflectors to the plane of motion be exactly 45°, let the index CP be directed to 180°: if the inclination of the reflectors to each other be not then = 90°, a correction muſt be applied. It will be known whether the inclination of the reflectors to each other be = 90°, by obſerving the two oppoſite horizons at ſea, and at land by various obvious methods. Theſe examinations are

are wholly independent of the inclination of the telescope to the plane of motion, which is regulated to its true situation parallel to the plane OPA, by making the plane OPA fixed in regard to distant objects, and by observing if the images of objects E, seen after two reflections of the rays, describe the arc of a great circle while C is carried round the plane of motion. Any three fixed objects, at a sufficient distance, and situated in the same plane with the observer's eye, will be sufficient for making this adjustment.

Fig. 9. 10. and 11. represent the progress of the rays, and the position of the reflectors according to this construction. TC is a ray issuing from any object T in a direction parallel to the plane of the motion, and impinging on the speculum C, which is inclined to that plane at an angle of 45°: from hence it is reflected in the direction CB perpendicular to the plane OCA, and being there reflected by the speculum B proceeds in the given or constant direction BG parallel to the plane of motion OPA.

16. There is another construction which follows from the general theory, the description of which should not be omitted. This will require some little explanation. As before (fig. 12.) let OPA represent the plane of motion, K its pole, FIM a parallel or lesser circle projected on it, the distance of this parallel from the pole K being measured by the arc FK; let KO be the primitive secondary, and BFD the fixed plane of reflection on the speculum B coincident with it. The other parts of the construction * remaining, it has been demonstrated (art. 10), that the sine of half the observed angle, that is, the sine of half ED, is proportional to the sine of IF, and the cosine of DL jointly. Every thing else being the same, it is manifest, that the sine of half ED will be proportional to the sine of IF: as therefore the arc KF, that is, the inclination of the reflecting planes to the plane of motion, is decreased,

* Compare fig. 2. and 3.

the

the angle meafured or arc ED will become fmaller at the fame time, becaufe FI decreafes with KF, the angle IKF remaining. This property feems applicable to good purpofe in meafuring fmall angles, not only from the great extent of fcale, which is here obtained, but from various advantageous circumftances, which will appear in the fubfequent article, and from the computations annexed in thofe which follow.

In this conftruction the fixed plane of reflection is made coincident with the primitive fecondary for various reafons: there are only two pofitions of that fixed plane which admit of eafy and exact adjuftments; thefe are when the fixed plane of reflection is either perpendicular (art. 11.) to the primitive fecondary or coincident with it. The latter pofition is preferred exclufive of the advantages it poffeffes in common with the other, becaufe it affords means for a very precife adjuftment of the inclination of the reflecting planes to the plane of motion, that is, of the arc KF; for if the primitive fecondary OKMD (fig. 12.) be produced, and in it DG be taken equal to four times KF, it is manifeft, that when the perpendicular CI coincides with M, or, which is the fame thing, when the radius CP is directed to 180°, the object E obferved by two reflections will coincide with G, becaufe BF = FD and BM = MG by conftruction. If then two given objects be obferved when the index points to 180°, the inclination of the plane reflectors to the plane of motion will be one fourth part of the angle fubtended by thefe objects.

Concerning the magnitude of the arc FB, being the meafure of the angle of incidence on the fixed fpeculum, and of KF = the inclination of the reflectors to the plane of motion, it will appear, by the computations*, that the fmaller they are both taken, every thing elfe being the fame, the more exact will be the refult of the obfervation; but both are limited by circumftances which

* Infra.

fhould

ſhould next be deſcribed; theſe will be more obvious if an outline be annexed, repreſenting this particular caſe of the theory adapted to the menſuration of ſmall angles when reduced to practice. OAPC (fig. 13.) is the plane of motion, C the moveable ſpeculum carried round in the plane of motion by the radius CP; a ray coming from any diſtant object T impinges on the ſpeculum C, and being reflected in the direction CB, is there again reflected in the direction BG, paſſing along the axis of a teleſcope. A ray coming from another diſtant object S, inclined to the ray TC at a ſmall angle enters the teleſcope parallel to the direction of its axis, which is coincident with BG, and conſequently the images of the two objects S and T will be ſeen to coincide in the middle of the field of the teleſcope, the angle ſubtended by them being BGT, which muſt be determined by the ſubſequent computations.

17. By this figure (fig. 13.), without further argument, it is plain, that the magnitude of the reflecting ſpeculum C limits the conſtant angle of incidence on B; for were that angle $= o$, the lines CB, BG, would coincide, by which means the ray BG and others adjacent to it would be intercepted from entering the teleſcope. The magnitude of the reflecting plane depends on the quantity of light required; if a circle of about 1.2 inches diameter be ſufficient, and the perpendicular diſtance of the reflecting planes be made equal to five inches, the leaſt angle of incidence, conſiſtent with theſe conditions, will be about 7°. It is however to be remembered, that the magnitude of the reflector's ſhould be adapted to the aperture of the teleſcope uſed in the obſervation. As the area of the ſpeculum increaſes, the light admitted into the ſame teleſcope decreaſes, and theſe areas ſhould be ſo proportioned as to afford equal quantities of light, ſo that the objects ſeen by two reflections, and by direct rays, may be nearly of equal brightneſs; but for the

fake of conftructing an example to this theory, the magnitude of the reflecting planes and the angle of incidence on the fixed fpeculum B depending on it may be affumed of the value mentioned in this article.

18. The magnitude of the arc KF, or of the inclination of the reflecting planes to the plane of motion is limited by the angles which the obferved objects fubtend (art. 16.). Fig. 12. Becaufe the greateft angle obfervable will be meafured by four times the arc KF, it follows, that the arc KF muft not be lefs than one fourth part of the greateft angle intended to be obferved by this conftruction; if the inclination denoted by the arc KF be fixed at 10′, four times that angle being 40′ will be greater than the apparent diameters of the fun or any of the planets.

19. It remains to infer from the preceding conftruction (fig. 2), the actual meafure of the angle fubtended by the objects obferved. This muft be effected by computation, which will not only ferve as an illuftration of the theory, but afford means of eftimating and comparing the errors in the angle deduced, occafioned by the unavoidable errors in obfervation and practical conftruction; an examination extremely ufeful in aftronomical fubjects: next to removing errors entirely from obfervations, which is fcarcely to be hoped for, the leffening, circumfcribing, and reducing them within known limits is an object of principal confequence.

20. The conftruction of fig. 2. remaining, through the points F and I (fig. 14.) draw the great circle FI. Bifect FI in Q, and through the points K and Q draw the arc KQ, which will be perpendicular to IF. To determine by computation the arc ED which meafures the angle fubtended by the obferved objects, three fpherical triangles, KQF or KIF, IFB, and DBE, muft be folved, for which the data are evidently fufficient:

or

or the value of ED may be obtained from the solution of two triangles KQF and DFL, with the proportion demonstrated in art. 9.

21. To proceed with the computation, through D draw the arc DL perpendicular to FI, and let the sine of QKF $= p$, being the sine of half the arc OP, the measure of IKF: put the sine of KF $= s$, the sine of DF $= m$, the sine of DFK $= n$, radius $= 1$. In the right-angled spherical triangle KQF, the properties of spherics give this proportion: as radius to the sine of KF so is the sine of QKF to sine of QF; wherefore sin. QF $= sp$; cos. QF $= \sqrt{1 - s^2p^2}$; and sin. FI (FI being double to QF) $= 2sp \times \sqrt{1 - s^2p^2}$. Moreover, because as rad. to cos. QK so is cos. QF to cos. KF, we have cos. KQ $= \frac{\sqrt{1-s^2}}{\sqrt{1-p^2s^2}}$; and sin. KQ $= \frac{\sqrt{s^2 - s^2p^2}}{\sqrt{1 - s^2p^2}}$. And since as rad.: sin. QFK so is sin. KF to sin. KQ; this proportion gives sin. QFK $= \frac{\sqrt{1-p^2}}{\sqrt{1-s^2p^2}}$; and because the sine of the angle LFD is the sine of the difference (or sum) of the angles QFK, DFK, of which the sines are, sin. QFK $= \frac{\sqrt{1-p^2}}{\sqrt{1-s^2p^2}}$ just found, and sin. DFK $= n$ by the data, we have from the rules of trigonometry,

$$\text{sin.}^* \text{ DFL} = \frac{\sqrt{1-p^2} \times \sqrt{1-n^2} \mp \sqrt{p^2n^2 - n^2s^2p^4}}{\sqrt{1-s^2p^2}},$$

and since in the right-angled triangle LDF, as rad.: sin. DF :: sin. DFL : sin. DL, and by the problem sin. DF $= m$ it appears, that

* If the points P and Q be on different * sides of the point O as they are represented in the construction, the last term will be affected with the sign $-$: if P and Q be on the same side of O, the sign of the last term will be $+$. It may be here observed, concerning the geometrical construction (fig. 2. and 3) that when P and Q are on different sides of O, the angle observed ED will be greater than when those points are on the same side of the initial point O, the arcs OP, Op, being equal.

* Compare fig. 2.

sin.

$$\text{fin. DL} = \frac{\sqrt{1-p^2} \times \sqrt{m^2 - n^2n^2} \mp \sqrt{p^2m^2n^2 - p^2n^2m^2s^2}}{\sqrt{1-s^2p^2}}, \text{ and}$$

$$\overline{\text{fin. DL}}^2 = \frac{m^2 - n^2n^2 - p^2m^2 + 2p^2m^2n^2 - m^2n^2p^2s^2 \mp 2m^2pn \times \sqrt{1-p^2} \times \sqrt{1-s^2} \times \sqrt{1-n^2}}{1-s^2p^2}$$

and the fquare of the cofine of DL

$$= \frac{1 - s^2p^2 - m^2 + m^2n^2 + p^2m^2 - 2p^2m^2n^2 + m^2n^2p^2s^2 \pm 2m^2pn \times \sqrt{1-p^2} \times \sqrt{1-s^2} \times \sqrt{1-n^2}}{1-s^2p^2}$$

The fine of IF was fhewn to be $2sp \times \sqrt{1-s^2p^2}$, and its fquare $= 4s^2p^2 \times \overline{1-s^2p^2}$: moreover, it was demonftrated in art. 9. that as $\overline{\text{rad.}}^2 : \overline{\text{cof. DL}}^2 :: \overline{\text{fin. IF}}^2 : \overline{\text{fin. } \tfrac{1}{2}\text{ED}}^2$ which gives, by fubftituting the values of $\overline{\text{cof. DL}}^2$ and $\overline{\text{fin. IF}}^2$, and multiplying the $\overline{\text{cof. DL}}^2$ into $\overline{\text{fin. IF}}^2$, $\overline{\text{fin. } \tfrac{1}{2}\text{ED}}^2 =$

$$4s^2p^2 \times \overline{1 - s^2p^2 - m^2 + m^2n^2 + p^2m^2 - 2p^2m^2n^2 + m^2n^2s^2p^2 \pm 2m^2pn \times \sqrt{1-s^2} \times \sqrt{1-p^2} \times \sqrt{1-n^2}}$$

and the cofine of $\tfrac{1}{2}\text{ED}^2 =$

$$1 - 4s^2p^2 \times \overline{1 - s^2p^2 - m^2 + m^2n^2 + p^2m^2 - 2p^2m^2n^2 + m^2n^2s^2p^2 \pm 2m^2pn \times \sqrt{1-s^2} \times \sqrt{1-p^2} \times \sqrt{1-n^2}}$$

finally, the cofine of ED is therefore =

$$1 - 8s^2p^2 \times \overline{1 - s^2p^2 - m^2 + m^2n^2 + p^2m^2 - 2p^2m^2n^2 + m^2n^2s^2p^2 \pm 2m^2pn \times \sqrt{1-s^2} \times \sqrt{1-p^2} \times \sqrt{1-n^2}}.$$

22. The particular cafes inferred from the geometrical conftruction may be compared with this analytical value of the cofine of ED, or of the angle fubtended by the obferved objects. If $s = 1$ and $n = 1$, by fubftituting 1 for s and n in the expreffion juft found, we fhall have the cofine of ED $= 1 - 8p^2 + 8p^4$, which is the cofine of an arc four times greater than that of which the fine $= p$. This anfwers to the properties of HADLEY's inftrument, in which KF or the inclination of the reflecting planes to the plane of motion is 90°, and its fine $= 1 = s$: moreover, in HADLEY's inftrument, the fixed plane of reflection at the unmoved fpeculum is parallel to the plane of motion, and therefore perpendicular to any fecondary of that plane; its inclination to any fecondary

condary therefore will be 90°, and the ſine of this inclination $= 1 = n$ by the problem. And ſince p is the ſign of half the inclination of the reflectors, the angle of which the coſine is $1 - 8p^2 + 8p^4$ will be twice the inclination of the reflecting planes, which is a property of HADLEY's inſtrument. In the analytical value of the coſine of ED, the laſt term is affected by two ſigns; theſe depend on the poſition of the ſecondary KP and the interſection Q in reſpect of the point O. If the ſecondary KP or the index CP be on the ſame ſide of O with the interſection Q (fig. 2.), the ſign of the laſt term is negative: if CP and Q be on oppoſite ſides of O, the ſign of the laſt term will be poſitive; and when $DFK = o$ or 180°, the whole term vaniſhes, becauſe in that caſe $n = o$. Alſo, if $m = 0$, $n = 1$, $s = 1$, or if $p = 1$, the laſt term vaniſhes. When $m = s = \frac{1}{\sqrt{2}}$, $KF = 45°$: in this caſe, if $n = o$ the conſtruction will be that deſcribed in art. 15. and the coſine of the obſerved angle ED will equal $1 - 2p^2$, the other terms vaniſhing: and becauſe $1 - 2p^2$ is the coſine of an arc double to that of which the ſine $= p$, it follows, that the angle obſerved will be equal to the arc deſcribed by the index from o, of which the ſine of one half is by the problem $= p$. In every caſe, when $n = o$, that is, when the fixed plane of reflection at the unmoved ſpeculum coincides with the primitive ſecondary KO (fig. 2. and 12.), the coſine of $ED = 1 - 8s^2p^2 \times \overline{1 - s^2p^2 - m^2 + p^2m^2}$.

23. The ſine of ED will be neceſſary (art. 27.) to aſcertain the variation of ED from the truth occaſioned by errors in the data; to obtain ſin. ED let

$1 - s^2p^2 - m^2 + m^2n^2 + p^2m^2 - 2p^2m^2n^2 + m^2n^2s^2p^2 \pm 2m^2np \times \sqrt{1-s^2} \times \sqrt{1-p^2} \times \sqrt{1-n^2}$ $= d$: then (art. 21.) from the value of $\overline{\text{coſ.}\ \tfrac{1}{2}ED}^2$ we have ſin. $ED = 4sp \times \sqrt{d} \times \sqrt{1 - 4s^2p^2d}$. When s is very ſmall, and

and $n=0$, $d=1-m^2+m^2p^2$ nearly, which gives

$$\text{fin. ED} = 4sp \times \sqrt{1-m^2+m^2p^2} \text{ nearly.}$$

24. The cofine of the obferved angle reprefented by ED (fig. 2. and 14.) in the conftruction, being computed from the four given quantities p, s, m, and n, if either of thefe fhould deviate from its true value, the angle deduced will be erroneous: and from the general expreffion for the cofine of ED, an eftimation of this error will be obtained. In the inveftigation, however, it muft be obferved, that although the fmall increments or decrements of arcs or fines are affumed proportional to the fluxions of thefe quantities, which is ftrictly true only in the nafcent ftate of the increments or decrements, yet when the given variations are in a practical fenfe very fmall, the eftimation of correfponding variations will be in general fufficiently exact for practical purpofes.

25. Small increments and decrements, that is, fmall variations, being affumed proportional to the fluxions of arcs and of their fines and cofines, if the variation of the fine or cofine of any given arc be known, the cotemporary variation of the arc will be for the moft part inferred from the following proportions: as fin. : rad. :: $-\dot{\overline{\text{cof.}}}$: $\dot{\overline{\text{arc}}}$; and as cof. : rad. :: $\dot{\overline{\text{fin.}}}$: $\dot{\overline{\text{arc.}}}$ But thefe proportions muft be ufed under reftrictions very neceffary to be inferted in this place, being true when applied to the intermediate parts of the quadrant only and failing at the extremities; for example, at the very beginning of the quadrant, or at the very end of the femi-circle, the variation of the cofine is the verfed fine of the arcs increment or decrement, which gives the proportion as fin. : 2 × rad. :: $-\dot{\overline{\text{cof.}}}$: $\dot{\overline{\text{arc}}}$, being wholly different from the former: in like manner, at the very extremity of the quadrant,

drant, the increment of the ſine becomes the verſed ſine of the arcs laſt increment, which gives this proportion: as coſ. : 2 × rad. :: $\dot{\overline{\text{ſin.}}}$: $\dot{\overline{\text{arc.}}}$ And ſince in this caſe $\dot{\overline{\text{arc}}}$ = coſine, we ſhall have, $\dot{\overline{\text{arc}}} = \sqrt{2 \times \dot{\overline{\text{ſin.}}}}$ radius being = 1. In the other parts of the quadrant which are not very near its extremity, $\dot{\overline{\text{arc}}} = \frac{\dot{\overline{\text{ſin.}}}}{\text{coſ.}}$; having given, therefore, the variation of the ſine or coſine of any arc, the ſine or coſine being known, the cotemporary variation of the arc itſelf may be obtained, when it is either at the very extremities of the quadrant; or at ſome diſtance from thoſe extremities. The difficulty lies in aſcertaining in what part of the quadrant the value of the $\dot{\overline{\text{arc}}} = \frac{\dot{\overline{\text{ſin.}}}}{\text{coſ.}} = \frac{-\dot{\overline{\text{coſ.}}}}{\text{ſin.}}$ begin to fail, and the value expreſſed by $\dot{\overline{\text{arc}}} = \sqrt{2 \times \dot{\overline{\text{ſin.}}}}$ or $-\sqrt{2 \times \dot{\overline{\text{coſ.}}}}$ to take place. This leads to a general propoſition comprehending both theſe values for the arc's variation, extended to every part of the quadrant.

The propoſition is this: the difference of the coſines is to the chord of the difference of any two arcs, as the ſine of an arithmetical mean between them to radius; and the difference of the ſines is to the chord of the difference, as the coſine of the ſame arithmetical mean to radius. Let AB, AF (fig. 15.) be the given arcs; BF their difference; BL, FH, the ſines; CL, CH, the coſines of the arcs AB, AF, reſpectively; join CA, CB, CF, and FB; FB will be the chord of the difference of the arcs AF, AB. Through B draw BG parallel to CA; then HL = BG will be the difference of the coſines, and FG the difference of the ſines. Biſect FB in D, ſo ſhall DA be an arithmetical mean between the arcs FA, BA; join DC, which will interſect FB at right angles in E; through D and E draw DK, EI, perpendicular

to

to CA: DK will be the fine, and CK the cofine of the mean arithmetical DA: the fimilar triangles CEI, CDK, FGB, give the following proportions:

HL or GB : FB :: DK : DC, and

GF : FB :: CK : DC, which was the propofition to be demonftrated *.

* When FB (fig. 15) is fo fmall in comparifon of FA, that FG fhall be evanefcent in comparifon of FH, FH and BL will be in the ratio of equality, and confequently the ratio FH : FC equal to the ratio BL : BC, or to the ratio DK : DC; for this reafon, and becaufe it has been proved, that as HL : FB :: DK : DC, it follows, that as HL : FB :: FH or BL : BC, that is, as the variation of the cofine is to cotemporary variation of the arc, fo is the fine of the varying arc to radius; and, for fimilar reafons, as the variation of the fine is to the cotemporary variation of the arc, fo is the cofine to radius.

If BA be fo diminifhed that FG fhall bear a finite proportion to FH, and too great to be neglected, BL will not be either to FH or to DK in a ratio of equality: confequently, FH or BL muft no longer be fubftituted for DK: as BA becomes lefs, FB being ftill fuppofed evanefcent, DK approximates to the fine of $\frac{1}{2}$FB to which it is ultimately equal when B and F arecoinciding with A (fig. 16.). In which cafe the proportion will become as HL or HA : FB or FA :: $\frac{1}{2}$FA : to CA, that is, as the verfed fine of FA is to the arc FA fo is half the arc FA to radius, or fo is the arc FA to diameter.

The propofitions which have been demonftrated, comprehend the variation of the arc expreffed in terms of the cotemporary variation of the fine or cofine in every part of the quadrant without limitation, it being only allowed to fubftitute the arc FB inftead of its chord, thefe quantities approximating the more nearly to equality as FB is fmaller, and being ultimately equal in their evanefcent ftate. Moreover, it will be eafy from what has preceded to conftruct a plane right-lined triangle, which fhall be fimilar to the mixtilinear triangle contained under an arc, its fine and verfed fine when they are diminifhed fine limite. Let FA (fig. 16.) be any arc, FA the chord, FH the fine, CH the cofine of the arc FA. Bifect FA in D, join CD, and draw the right fine DK: then will the plane right-lined triangle KDC continually approximate to fimilarity with the mixtilinear triangle FDAH as FA becomes fmaller, and the two triangles will be ultimately fimilar when FA is vanifhing.

By

From these geometrical proportions, having given any arc and the variation of its sine or cosine, the cotemporary variation of the arc may be estimated by computation in general for any part of the quadrant. Let the sine of any arc be s, the cosine $= c$, the chord of the arc's variation $= x$, the given variation of the cosine $= d$, or the given variation of the sine $= b$, radius $= 1$; then if the cosine of the arc increases by the difference d, the chord of the cotemporary decrease of the arc, or

$$-x = \sqrt{2s^2 - 2dc \mp \sqrt{\overline{2s^2 - 2dc}^2 - 4d^2}}$$

and if the sine of the given arc increases by the difference b

$$+x = \sqrt{2c^2 - 2bs \mp \sqrt{\overline{2c^2 - 2bs}^2 - 4b^2}},$$

which are the mathematically true values of the chord FB, and will approximate to the magnitude of the arc FB as that arc is continually diminished. The following expressions for the chord of the variation x are more compendious, and will be sufficiently near the truth when FB is very small.

$$-x = \frac{s}{c} - \sqrt{\frac{s^2}{c^2} - \frac{2d}{c}}$$

$$+x = \frac{c}{s} - \sqrt{\frac{c^2}{s^2} - \frac{2b}{s}}.$$

In these four expressions it must be observed, that the sine and cosine are supposed to vary by increase: should the variation be a decrement, the sign of x and of b or d must be changed.

26. Let the quantities p, s, m, n, vary by small increments $\dot{p}$, $\dot{s}$, $\dot{m}$, $\dot{n}$, respectively, then to obtain the cotemporary variation of cos. ED, because (art. 21.)

$$\text{Cos. ED} = 1 - 8s^2p^2 \times \overline{1 - s^2p^2 - m^2 + m^2n^2 + m^2p^2 - 2p^2m^2n^2 +}$$
$$\overline{m^2n^2s^2p^2 \pm 2m^2np \times \sqrt{1-p^2} \times \sqrt{1-n^2} \times \sqrt{1-s^2}},$$

by taking the fluxion of the equation we have

$$\overline{\text{Cof. }\dot{\text{ED}}} = -16s^2p\dot{p} \times \overline{1 - 2s^2p^2 - m^2 + n^2m^2 + 2m^2p^2 - 4m^2n^2p^2 + 2n^2m^2s^2p^2 \pm \frac{m^2np \times \sqrt{1-n^2} \times \sqrt{1-s^2} \times \overline{3-4p^2}}{\sqrt{1-p^2}}}$$

$$-16p^2s\dot{s} \times \overline{1 - 2s^2p^2 - m^2 + n^2m^2 + p^2m^2 - 2p^2m^2n^2 + 2p^2m^2n^2s^2 \pm \frac{m^2np \times \sqrt{1-n^2} \times \sqrt{1-p^2} \times \overline{2-3s^2}}{\sqrt{1-s^2}}}$$

$$+16s^2p^2m\dot{m} \times \overline{1 - n^2 - p^2 + 2p^2n^2 - s^2p^2n^2 \mp 2pn \times \sqrt{1-n^2} \times \sqrt{1-s^2} \times \sqrt{1-p^2}}$$

$$+16s^2p^2m^2\dot{n} \times \overline{-n + 2p^2n - s^2p^2n \mp \frac{\sqrt{1-s^2} \times \sqrt{1-p^2} \times p \times \overline{1-2n^2}}{\sqrt{1-n^2}}}.$$

27. This value of $\overline{\text{cof. }\dot{\text{ED}}}$ is expreſſed in terms of the variation of the ſines of the given quantities: if it be neceſſary to expreſs $\overline{\text{cof. }\dot{\text{ED}}}$ in terms of the variation of the arcs themſelves, it muſt firſt be conſidered to what part of the quadrant they belong: for example, if s be a ſine of an arc b not very near the extremity of the quadrant, and the variation be $\dot{s}$, the cotemporary variation of the arc $\dot{b}$ will be $\frac{\dot{s}}{\sqrt{1-s^2}}$; but if the variable arc be nearly $= 90°$, and becomes exactly equal to it ultimately having varied by a ſmall arc $\dot{b}$ of which the verſed ſine $= v$; then will $-\dot{s} =$ the verſed ſine of $\dot{b}$ and $-\dot{b} = \sqrt{2v}$. Laſtly, if the variable angle approximates to 90°, but is not equal to it, and the variation of its ſine ſhould be $= \dot{s}$, the cotemporary variation of the arc muſt be obtained from the general theorem in art 25. When either of the two latter caſes happen, the variation of the arc muſt be determined for each particular caſe; but it will be neceſſary to give a general expreſſion for $\overline{\text{cof. }\dot{\text{ED}}}$ in terms of the variations of the given arcs, of which p, s, m, n, are the reſpective ſines when theſe arcs are at ſome diſtance from 90°; this is contained in the next article.

28. Let

28. Let the angle QKF $= a$ (fig. 14.); the arc KF $= b$; the arc DF $= c$, and the angle DFK $= d$; their reſpective increments being $\dot{a}$, $\dot{b}$, $\dot{c}$, and $\dot{d}$, their ſines p, s, m, and n, and the cotemporary increments of their ſines $\dot{p}$, $\dot{s}$, $\dot{m}$, and $\dot{n}$: from the proportion contained in art. 24. we ſhall have $\dot{p} = \dot{a} \times \sqrt{1 - p^2}$, $\dot{s} = \dot{b} \times \sqrt{1 - s^2}$, $\dot{m} = \dot{c} \times \sqrt{1 - m^2}$, and $\dot{n} = \dot{d} \times \sqrt{1 - n^2}$, which being ſubſtituted in the value of $\overline{\text{coſ. } \dot{ED}}$ laſt found will give

$$\overline{\text{Coſ. } \dot{ED}} = -16s^2 \times \sqrt{1-p^2} \times p\dot{a} \times \overline{1 - 2s^2p^2 - n^2 + n^2m + 2m^2p^2 - 4m^2n^2p^2 + 2s^2m^2n^2p^2 \pm \frac{m^2np \times \sqrt{1-s^2} \times \sqrt{1-n^2} \times \overline{3-4p^2}}{\sqrt{1-p^2}}}$$

$$-16p^2 \times \sqrt{1-s^2} \times s\dot{b} \times \overline{1 - 2s\,p^2 - m^2 + n^2m^2 + p^2m^2 - 2p^2m^2n^2 + 2s^2m^2n^2p^2 \pm \frac{m^2np \times \sqrt{1-n^2} \times \sqrt{1-p^2} \times \overline{2-3s^2}}{\sqrt{1-s^2}}}$$

$$+16s^2p^2 \times \sqrt{1-m^2}\, m\dot{c} \times \overline{1 - n^2 - p^2 + 2p^2n^2 - s^2p^2n^2 \mp 2pn \times \sqrt{1-n^2} \times \sqrt{1-s^2} \times \sqrt{1-p^2}}$$

$$+16s^2p^2m^2 \times \sqrt{1-n^2} \times \dot{d} \times \overline{-n + 2p^2n - s^2p^2n \mp \frac{p \times \sqrt{1-s^2} \times \sqrt{1-p^2} \times \overline{1-2n^2}}{\sqrt{1-n^2}}}.$$

This quantity (art. 23.) being divided by the ſine of the obſerved angle, the variation of that angle or $\overline{\dot{ED}}$ will be the quotient.

29. In the expreſſion for $\overline{\text{coſ. } \dot{ED}}$ contained in art. 26. the variations $\dot{p}$, $\dot{s}$, $\dot{m}$, and $\dot{n}$, are arbitrary, as are $\dot{a}$, $\dot{b}$, $\dot{c}$, and $\dot{d}$, in the laſt article. If a condition be annexed to the variation of any of them, two or more may become dependant on each other; and their relation muſt be determined by the nature of the caſe. Moreover, if one or more of the given arcs and their ſines ſhould be correct, the variations correſponding and all the terms multiplied into them will vaniſh. To give an example of the uſe of theſe expreſſions before they are applied to the immediate purpoſe of examining the new conſtructions

defcribed in art. 15. and 16. let it be required to affign what error is occafioned in obferving a given angle with a HADLEY's fectant, in which the telefcope is parallel to the plane of motion, but the two reflectors deviate from their perpendicular to that plane by a fmall angle $\dot{b}$. Suppofe the error of half the arc pointed to by the index to be $\dot{a}$, and confequently the error of the fine of half that * arc $= \dot{a} \times \sqrt{1-p^2} = \dot{p}$: in this cafe, becaufe the inclination of the reflectors to the plane of motion is nearly $= 90°$, the variation of the fine will be equal to the verfed fine of the fmall arc $\dot{b}$, by which the inclination deviates from $90°$; let v be the verfed fine of $\dot{b}$, then will $-\dot{s} = v$ (s varying by a decrement of v). Moreover, becaufe a condition is annexed, which is, that the line of obfervation is parallel to the plane of motion, the variations $\dot{s}$, $\dot{m}$, and $\dot{n}$, will be dependent on each other. To inveftigate their relation let $FO = \dot{b}$ (fig. 7.) be the fmall arc which meafures the deviation of the reflectors from the perpendicular to the plane of motion: then, becaufe fin. DO $= m$, and fin. DOF $= n = 1$ by the problem, when F from having been coincident with O has moved through the arc OF, it is plain, that $\dot{n} =$ fin. DFO − fin. DOF = ($\dot{n}$ being a decrement); but, by the † properties of fpherics, cof. DFO $= \frac{\sqrt{2v} \times \text{cof. DF}}{\text{fin. DF}} = \frac{\sqrt{2v} \times \sqrt{1-m^2}}{m}$: and FO being very fmall, the

* p here, as in the general folution, denotes the fine of half the arc to which the index on the plane of motion is directed, that is, $p =$ the fine of one-fourth of the angle obferved in Mr. HADLEY's conftruction.

† Fig. 7. as rad. : cotang. DF :: tang. FO : cof. DFO, that is, FO being very fmall, and therefore $\overline{FO}^2 = 2 \times$ verfed fine of FO, as rad. : cotang. DF :: $\sqrt{2v}$: cof. DFO : by the problem fin. DF $= m$, and cof. DF $= \sqrt{1-m^2}$, wherefore cot. DF $= \frac{\sqrt{1-m^2}}{m}$, which gives cof. DFO $= \sqrt{2v} \times \frac{\sqrt{1-m^2}}{m}$.

fine

sine of DFO $= 1 - \frac{v \times \overline{1-m^2}}{m^2}$, from which $1 =$ sin. DOF being subtracted leaves $\dot{n} = \frac{-v \times \overline{1-m^2}}{m^2}$; or because $\dot{s} = -v$, $\dot{n} = \frac{+\dot{s} \times \overline{1-m^2}}{m^2}$. Moreover, the* quantity $\frac{1-s^2}{1-n^2}$ in the nascent state of $\overline{1-s^2}$ and $\overline{1-n^2} = \frac{m^2}{1-m^2}$: and $\sqrt{\frac{1-s^2}{1-n^2}} = \frac{m}{\sqrt{1-m^2}}$. Making therefore in the general expression contained in art. 26. $\frac{-v \times \overline{1-m^2}}{m^2} = \dot{n}$, $-v = \dot{s}$, $\dot{a} \times \sqrt{1-p^2} = \dot{p}$, and $\frac{m}{\sqrt{1-m^2}} = \sqrt{\frac{1-s^2}{1-n^2}}$, $n = s = 1$, we shall have

$\text{Cos.}\,\dot{\text{ED}} = -16p\dot{a} \times \sqrt{1-p^2} \times \overline{1-2p^2} + 16p^2v \times \overline{1-2p^2+p^2m^2 \mp mp \times \sqrt{1-m^2} \times \sqrt{1-p^2}}$ $+ 16p^2v \times \overline{1-p^2-m^2+m^2p^2 \mp mp \times \sqrt{1-m^2} \times \sqrt{1-p^2}} = -16p\dot{a} \times \sqrt{1-p^2} \times \overline{1-2p^2} +$ $16p^2v \times \overline{2-3p^2-m^2+2p^2m^2 \mp 2mp \times \sqrt{1-m^2} \times \sqrt{1-p^2}}$: and because the sine of the observed angle is $4p \times \sqrt{1-p^2} \times \overline{1-2p^2}$, the error of the observation itself, that is,

$$\dot{\text{ED}} = 4\dot{a} - \frac{4pv \times \overline{2-3p^2-m^2+2p^2m^2 \mp 2mp \times \sqrt{1-m^2} \times \sqrt{1-p^2}}}{\sqrt{1-p^2} \times \overline{1-2p^2}}.$$

In this example the position of the telescope has been supposed exactly in the plane of motion; should it be inclined to that plane at a small angle, of which the versed sine $= v$, the position of the reflectors and the arc pointed to by the index being correct, the general value of $\text{cos.}\,\dot{\text{ED}}$ will give the error of the observation, or $\dot{\text{ED}} = \frac{-4pv \times \sqrt{1-p^2}}{1-2p^2}$ †.

30. To

* The nascent value of $\frac{1-s^2}{1-n^2} = \frac{-2s\dot{s}}{-2n\dot{n}}$; but $\dot{n} = \frac{-v \times \overline{1-m^2}}{m^2} = \frac{+\dot{s} \times \overline{1-m^2}}{m^2}$: wherefore $\frac{1-s^2}{1-n^2} = \frac{m^2}{1-m^2}$, when s and n are nearly $= 1$.

† When the position of the telescope only is erroneous, the points F and O coincide

30. To examine in what degree an obfervation taken by the new conftruction defcribed in art. 15. is affected by known errors in the given quantities, let the reflectors B and C deviate by excefs from their true angle of inclination to the plane of motion by a fmall angle $+\dot{b}$: let the angle of incidence on the fixed fpeculum be too great by the increment $\dot{c}$: let the fixed plane of reflection deviate from the fecondary KO with which it fhould coincide by a fmall angle $\dot{d}$; and laftly, let the error of the arc pointed to by the index be $= 2\dot{a}$; then thefe variations are arbitrary, no condition being annexed. Moreover, by the conftruction $m = s = \frac{1}{\sqrt{2}}$, and $n = o$, which values being fubftituted in the general expreffion contained in art. 28. we fhall have.

$$\overline{\text{Cof.}\dot{\text{ED}}} = -4\dot{a}p \times \sqrt{1-p^2} - 4p^2\dot{b} \times \overline{1-p^2} + 4p^2\dot{c} \times \overline{1-p^2} \mp 2\sqrt{2}p^3\dot{d} \times \sqrt{1-p^2};$$

and becaufe the fine of the angle meafured $= 2p \times \sqrt{1-p^2}$, the error of the obfervation required, or

coincide (fig. 7.) let the inclination of the telefcope to the plane of motion with which it fhould coincide, be meafured by the fmall arc Dd; then the correfponding variation of the angle DOK will be DOd. Let D$d = e$, and its verfed fine $= v$; fince the fine of DO $= m$, and the fine of DOK $= 1 = n$ by the problem, $\dot{n} =$ the verfed fine of DOd; but DO$d = \frac{e}{m}$, and the verfed fine of DO$d = \frac{e^2}{2m^2} = \frac{v}{m^2}$: wherefore $\dot{n} = \frac{-v}{m^2}$. This being premifed, it appears from art. 26. when $\dot{p}$, and $\dot{s}$, are $= o$, that $\overline{\text{cof.}\ \dot{\text{ED}}} = +16s^2p^2m^2\dot{n} \times \overline{-n + 2p^2n - s^2p^2n}, \pm \frac{\sqrt{1-s^2} \times \sqrt{1-p^2} \times s \times \overline{1-2n^2}}{\sqrt{1-n^2}}$

$$+ 16s^2p^2m\dot{m} \times \overline{1 - n^2 - p^2 + 2p^2n^2 - s^2p^2n^2 \mp 2pn \times \sqrt{1-n^2} \times \sqrt{1-s^2} \times \sqrt{1-p^2}}$$

in which quantity, fubftituting 1 for s, 1 for n, and $\frac{-v}{m^2}$ for $\dot{n}$, we fhall have $\overline{\text{cof.}\ \dot{\text{ED}}} = +16p^2v \times \overline{1-p^2}$, which being divided by the fine of the obferved angle $= 4p \times \sqrt{1-p^2} \times \overline{1-p}$, the quotient will be the variation of that angle or $\dot{\text{ED}} = \frac{-4vp \times \sqrt{1-p^2}}{1-2p^2}$.

ED =

$$\dot{\overline{ED}} = 2\dot{a} + 2p\dot{b} \times \sqrt{1-p^2} - 2p\dot{c} \times \sqrt{1-p^2} \pm \sqrt{2p^2}\dot{d}.$$

It appears from the firſt term $2\dot{a}$, that an error in the arc pointed to by the index, cauſes an equal error in the obſerved angle; whereas a double error is cauſed by it in HADLEY's ſextant, which gives the new conſtruction conſiderable advantages: to counter-balance theſe the errors in HADLEY's conſtruction, cauſed by a wrong poſition of the reflecting planes, &c. are almoſt evaneſcent; whereas the three laſt terms in the value of $\dot{\overline{ED}}$ juſt found, may become conſiderable, unleſs great care be taken in making the adjuſtments: the ſeparation alſo of the images, when in contact, cauſed by any unſteadineſs of the inſtrument will be greater than in Mr. HADLEY's conſtruction *; but

* The variation of the obſerved angle $\dot{\overline{ED}}$ will ſhew how much the images ſeen in contact in the field of the teleſcope will appear to divaricate on any motion of the entire conſtruction. For example, while the images of the obſerved objects are coincident in the field of the teleſcope, ſuppoſe that Mr. HADLEY's inſtrument were turned round in its own plane through a ſmall angle: here s, n, and p, not being affected by this motion, it follows, that $\dot{s}$, $\dot{n}$, and $\dot{p} = 0$, $m =$ the ſine of the angle of incidence on the fixed ſpeculum, being the only quantity which ſuffers alteration: let its variation $= \dot{m}$, which will give from art. 26. the correſponding variation in the coſine of the obſerved angle, or

$\text{Cof.}\,\dot{\overline{ED}} = +16s^2p^2m\dot{m} \times \overline{1-n^2-p^2+2p^2n^2-s^2p^2n^2 \mp 2pn \times \sqrt{1-n^2} \times \sqrt{1-s} \times \sqrt{1-p^2}}$ $= 0$, becauſe $n = s = 1$. Wherefore any motion of the images in the plane of the inſtrument will not cauſe the leaſt ſeparation of them. Now ſuppoſe the whole to be turned on an axis ſituated in the plane of motion, and perpendicular to the teleſcope's axis: if the angular motion be meaſured by an arc of which the verſed ſine $= v$, the points in contact will be ſeparated through an angle $= \frac{4vp \times \sqrt{1-p^2}}{1-2p^2}$, p being the ſine of one quarter of the angle obſerved; but the quantity v being very ſmall, when the angular motion does not exceed 30′, the divarication of the images will be inconſiderable. All oblique motions of the teleſcope's axis, and conſequently of the image ſeen by direct rays, may be reſolved into thoſe that have

but in meafuring the fmaller angles, this feparation of the images, as well as the errors expreffed by the three laft terms will be greatly diminifhed while that which is denoted by $2\dot{v}$ contained in the firft term is not increafed. On the whole, from the properties which have been demonftrated to belong to this conftruction defcribed in art. 15. it may feem worthy of attention in practice, for fome aftronomical as well as other ufes.

31. By the fame way of examination it may be judged, whether the method of obferving by two reflections from plane furfaces be applicable to the menfuration of fmall angles, according to the conftruction defcribed in art. 16. Let the errors of the four given quantities (as in the laft article) be $2\dot{a}$ = the error of the arc

have been confidered, which are perpendicular to each other: and from hence the reafon appears, why the motion of a fhip at fea does not much difturb the obfervation of angles by Mr. HADLEY's inftrument.

The new conftruction defcribed in art. 15. is not fo well adapted for obfervation where it cannot be fteadily fixed. When the images are in contact, if the inftrument be turned in its own plane through a fmall angle e'' the feparation of the images will be $= 2e''p^2$ (becaufe $d'' = \sqrt{2}\,e''$, vid. p. 426.) p fignifying the fine of half the obferved angle: this it is evident will moft affect the obfervation of the larger angles; but in meafuring thofe that are fmall, the divarication will become inconfiderable. Moreover, if the angular motion of the inftrument be e'', when it turns round an axis in the plane of motion, and perpendicular to the telefcope's axis, the feparation of the images will be $= 2e''p \times \sqrt{1-p^2}$, p. 426. which it is plain will moft difturb the obfervations of angles about 90°, but will fcarcely alter the contact of the images, when the angles meafured are very fmall, or near 180°.

The objects obferved and their images are here underftood to be phyfical points: thus, when the two images of the fun are feen by direct and reflected rays, and the limbs appear precifely in contact, if by any motion of the inftrument the contact is difturbed, the points which before touched, being the obferved objects, are faid to be feparated, whether the centres of the folar images approach or recede from each other, the feparation being eftimated in the direction of an arc which paffes through the centers of the two folar images.

Experience muft determine in what degree this feparation of the images will difturb obfervations taken at fea with the new conftruction.

pointed

pointed to by the index; $\dot{b}=$ the error of the inclination of the reflectors to the plane of motion; $\dot{c}=$ the error in the angle of incidence on the fixed ſpeculum; and $\dot{d}=$ the inclination of the fixed plane of reflection to the primitive ſecondary with which it ſhould coincide. Referring to the general value of $\overline{\text{col.}\,\dot{\text{E}}\text{D}}$, (art. 28.) and making $n=o$ we ſhall have

$$\overline{\text{Col.}\,\dot{\text{E}}\text{D}}=-16\times\sqrt{1-p^2}\,ps^2\dot{a}\times\overline{1-2s^2p^2-m^2+2m^2p^2}-16p^2s\dot{b}\times\sqrt{1-s^2}\times\overline{1-2p^2s^2-m^2+m^2p^2}$$
$$+16s^2p^2\dot{c}m\times\sqrt{1-m^2}\times\overline{1-p^2}\mp16m^2p^3s^2\dot{d}\times\sqrt{1-s^2}\times\sqrt{1-p^2},$$

and becauſe s being very ſmall, the ſine of ED (art 23.) approximates to $4sp\times\sqrt{1-m^2+p^2m^2}$, the error of the obſervation itſelf, or $\overline{\dot{\text{E}}\text{D}}=\dfrac{4\dot{a}s\times\sqrt{1-p}\times\overline{1-2p^2s^2-m^2+2p^2m^2}}{\sqrt{1-m^2+p^2m^2}}$

$$+\frac{4p\dot{b}\times\sqrt{1-s^2}\times\overline{1-2p^2s^2-m^2+m^2p^2}}{\sqrt{1-m^2+p^2m^2}}$$
$$-\frac{4spm\dot{c}\times\sqrt{1-m^2}\times\overline{1-p^2}}{\sqrt{1-m^2+p^2m^2}}$$
$$\pm\frac{4m^2p^2s\dot{d}\times\sqrt{1-s^2}\times\sqrt{1-p^2}}{\sqrt{1-m^2+p^2m^2}}.$$

The firſt term of this expreſſion gives the relation between any ſmall variation in the arc pointed to by the index, and the correſponding alteration in the angle obſerved; if therefore the variation on the divided arc be any ſmall angle $\pm2\dot{a}$, + or − the firſt term will expreſs the variation by which the obſerved angle is increaſed or diminiſhed. According to the magnitude of m and s aſſumed for this conſtruction m being the ſine of 7°, and $s=$ the ſine of 10′, it appears, that at the very beginning of the ſcale one ſecond of a degree in the angle obſerved correſponds to ſomewhat leſs than three minutes on the divided arc OP; that is, when $2\dot{a}=$ about 173″, $\overline{\dot{\text{E}}\text{D}}=1''$, $\dot{c}$, $\dot{b}$, and $\dot{d}$, not being here conſidered. When $p=\frac{1}{2}$, the index then pointing to 60°, one ſecond in the obſerved angle cor-

reſpods

refponds to about 199″ in the divided arc OP : when p is nearly $=1$, the index being then directed to almoft 180°, it muft defcribe above 2 degrees to make an alteration of 1″ in the obferved angle. The fecond term expreffes the variation in the obfervation occafioned by an error $\dot{b}$ in adjufting the inclination of the reflecting planes to the plane of motion; but $\dot{b}$ (art. 18.) cannot exceed $\frac{1}{4}$ of the leaft angle vifible in the telefcope, confequently the utmoft value of the fecond term cannot be fo great as that leaft angle, being at its limit when $p=1$: it is manifeft when p is fmall, that the fecond term is fo much diminifhed, as to be in a phyfical fenfe evanefcent. The fame may be faid of the fourth term, containing the error of the optical adjuftment $\dot{d}$, which befides is multiplied into s the fine of 10′. The third term is occafioned by the error $\dot{c}$, for which, confiderable latitude muft be allowed, fuppofe 3′: to eftimate the effect of this error on the obfervation, let a cafe be affumed: let the index be directed to 90° when an obfervation is taken for determining the angle fubtended between two objects: then will $p=\frac{1}{\sqrt{2}}$; by fubftituting $\frac{1}{\sqrt{2}}$ for p, the fine of 7° for m, the fine of 10′ for s, and 180″ for $\dot{c}$ in the third term, we fhall have, by computation, the value of that term, or the error in the obfervation occafioned by this deviation of the angle of incidence from its true magnitude $=.''090$ not the tenth part of a fecond. This is rather an unfavourable cafe, the variation being not much lefs than at its maximum when $p=\frac{1}{\sqrt{2}}$: if p is fmall, or nearly $=1$, the * variation will be wholly infenfible.

32.

* The variation is a maximum when $p=$ the fine of 35° 12′; and confequently the arc pointed to by the index $=70°$ 24′. Subftituting therefore the fine of 35° 12′ for

32. From contemplating the value of the error $= \overline{ED}$, as expressed by the four terms just referred to, some further observations are suggested. It appears, that setting aside the error of the scale expressed by the first term (the angles measured being here supposed less than 22′ 56″ 33‴) all the others become smaller as the quantity p, and consequently the angle measured is diminished, p being multiplied into each of the three last terms. This is a material circumstance: for the same given error would affect the mensuration of the smaller angles in a greater proportion than the larger. Moreover, if $p = 1$, or approximates to that quantity, the index then pointing to an angle nearly = to 180°, the first term and the two last terms become almost evanescent, not only from the circumstances that have been considered, but from the quantities $\overline{1-p^2}$, $\sqrt{1-p^2}$, which are multiplied into them. This also the observer may avail himself of: for supposing he should know, that the angle of incidence on the fixed speculum, and the fixed plane of reflection, are imperfectly adjusted, and even that the divided arc is incorrect, he may almost wholly avoid the errors which they would occasion, by so adjusting the inclination of the reflectors to the plane of motion, in respect of the angle measured, that

for p in the third term, (p. 429.) every thing else remaining, the maximum of variation in the observed angle on account of 3′ difference in the angle of incidence on the fixed speculum will appear to be $=$ ″.098.

The error of an observation occasioned by a variation $\dot{c}$ in the angle of incidence on the fixed speculum, can never exceed the $\frac{1}{1837}$ part of $\dot{c}$, which is the maximum of error, the angle measured being $=$ 22′ 56″ 33‴, the index then pointing to 70° 24′. The error, caused by a small angle $\dot{d}$, at which the fixed plane of reflection is inclined to the primitive secondary with which it should coincide, cannot exceed the $\frac{1}{14336}$ part of $\dot{d}$, which is its maximum when the angle measured is 32′ 33″ 3‴, the index then being directed to 109° 20′.

the

the index ſhall point to ſome degree near to 180°, which is done by making that inclination very little more than one-fourth of the angle to be obſerved.

33. The tables I. and II. are calculated for taking the diameters of the ſun and planets: the conſtruction being formed on the principles which have been explained (art. 16. 17. 18. 31. 32. fig. 12. 13.). The fixed plane of reflection is coincident with the primitive ſecondary, and conſequently $n=0$: the common inclination of the reflectors to the plane of motion = 10′; and the conſtant angle of incidence on the fixed ſpeculum = 7°.

TABLE I.

Arc pointed to by the index.	Obſerved angle. ′	″	‴	Diff. ″	‴
1°	0	20	47	20	47
2°	0	41	34	20	47
3°	1	2	21	20	47
4°	1	23	8	20	46
5°	1	43	54	20	46
6°	2	4	40	20	45
7°	2	25	25	20	45
8°	2	46	10	20	44
9°	3	6	54	20	43
10°	3	27	37		

TABLE II.

Arc pointed to by the index.	Obſerved angle. ′	″	‴	Diff. ″	‴
100°	30	32	51	13	28
101°	30	46	19	13	20
102°	30	59	39	13	12
103°	31	12	51	13	3
104°	31	25	54	12	54
105°	31	38	48	12	46
106°	31	51	34	12	37
107°	32	4	11	12	28
108°	32	16	39	12	19
109°	32	28	58	12	11
110	32	41	9		

The

The divarication of the images while they traverse the field of the telescope during the time of an observation, and the errors of the observed angle in consequence of any change in the quantities m * and n (p. 429) should they appear to be of sensible magnitude, may be diminished until they are in physical sense evanescent, by altering the values of s and m (art. 31.): for this purpose it will be requisite to use two separate constructions; the one for observing very small angles, those, for example, which do not exceed $2'$: and the other, for measuring such angles as are subtended by the sun and moon. In the former of these s may be assumed = the sine of $30''$, m = sin. $7°$, and $n = 0$. From these data tab. III. is calculated. In the other construction, because very small reflecting surfaces are necessary to observe the sun by two reflections (art. 17.) m may be assumed = sin. $1°$, s being = sin. $10'$, and $n = 0$: from which conditions the fourth table is calculated. These tables may be easily extended, by calculating from the value of cos. ED or $\overline{\text{sin. } \tfrac{1}{2}\text{ED}}^2$ contained in art. 21. or 22.

* Let the telescope with the entire construction be steadily fixed: then if the objects observed be in contact while the touching points occupy the center of the field, the angle of incidence on the fixed speculum will be of its true magnitude; that is, its sine or m will = sin. $7°$ in tables I. II. and III.; and m = sin. $1°$ in table IV. The fixed plane of reflection also will be coincident with the primitive secondary in all the constructions corresponding to tables I. II. III. and IV. The telescope continuing unmoved, the diurnal motion of the heavens will cause the points in contact to leave the center of the field; this will occasion no alteration in the quantities s and p, but will affect m and n only: therefore, $\dot{s}$ and $\dot{p} = 0$. To estimate the effects of this change in the values of m and n, let $\dot{m} = \dot{c} \times \sqrt{1 - m^2}$, and $\dot{n} = \dot{d} \times \sqrt{1 - n^2}$, as in art 28. and suppose a line to be drawn through the centre of the field in a plane, perpendicular to the plane of motion: this line will be in the fixed plane of reflection; and any deviation of the points in contact through a small angular space c, from the center of the field will cause an equal variation $\dot{c}$ in the

the angle of incidence on the fixed fpeculum; the correfponding feparation of the images will be $= \frac{4s p m \dot{c} \times \sqrt{1-m^2} \times \overline{1-p^2}}{\sqrt{1-m^2+m^2p^2}}$ (p. 429.) which is the greateft poffible when $p = \sqrt{\frac{\sqrt{9-10m^2+m^4}-3-3m^2}{4m^2}}$ = fine of 35° 11′ 47″ (if m = fin. 7°, and s = fin. 10′, as in tab. I. and II.) being then $= \frac{\dot{c}}{1837}$; but if s = fin. 30″, as in tab. III. the greateft feparation will be only $\frac{\dot{c}}{36740}$, correfponding to lefs than one-third of a degree for the images motion through 10′ in the field of the telefcope. In like manner let a line be drawn through the center of the field perpendicular to the line before defcribed; any deviation of the points in contact through a fmall angle $\dot{e}$ in the direction of this line will caufe the plane of reflection at the fixed fpeculum to be inclined to the primitive fecondary with which it fhould coincide at an angle $= \frac{\dot{e}}{m}$; this will occafion a feparation of the images $= \frac{4mp^2s\dot{e} \times \sqrt{1-s^2} \times \sqrt{1-p^2}}{\sqrt{1-m^2+m^2p^2}}$ (page 429. $\frac{\dot{e}}{m}$ being there fubftituted for $\dot{d}$) which is the greateft poffible when $p = \sqrt{\frac{\sqrt{9-8m^2}-3-4m^2}{4m^2}}$. If m = fin. 7° and s = fin. 10′, as in tab. I. and II. the greateft feparation of the images will be $= \frac{\dot{e}}{1828} = \frac{\dot{d}}{14996}$: but if s = fin. 30″ and m = fin. 7°, as in tab. III. it will only $= \frac{\dot{e}}{36560} = \frac{\dot{d}}{299920}$, which anfwers to a divarication of lefs than 1‴ for the images motion through 10′ in the field of the telefcope, in the direction of the line above defcribed, which is drawn through the center of the field, and perpendicular to the fixed plane of reflection. It muft be remembered, that the feparations of the images here eftimated are greater than can poffibly happen in thefe conftructions, when the index is directed to any other points of the circumference of the plane of motion (the diftance of the images from the center the field not exceeding 10′) and are, even in this cafe, phyfically fpeaking, evanefcent.

TABLE

TABLE III.

Arc pointed to by the index.	Observed angle.			Diff.
	′	″	‴	″
100°	1	31	39	
101°	1	32	19	40
102°	1	32	59	40
103°	1	33	39	40
104°	1	34	18	39
105°	1	34	56	38
106°	1	35	34	38
107°	1	36	12	38
108°	1	36	50	38
109°	1	37	27	37
110°	1	38	3	36

TABLE IV.

Arc pointed to by the index.	Observed angle.			Diff.	
	′	″	‴	″	‴
100°	30	38	23		
101°	30	51	47	13	24
102°	31	5	2	13	15
103°	31	18	9	13	7
104°	31	31	7	12	58
105°	31	43	56	12	49
106°	31	56	37	12	41
107°	32	9	9	12	32
108°	32	21	32	12	23
109°	32	33	46	12	14
110°	32	45	52	12	6

Fig. I.

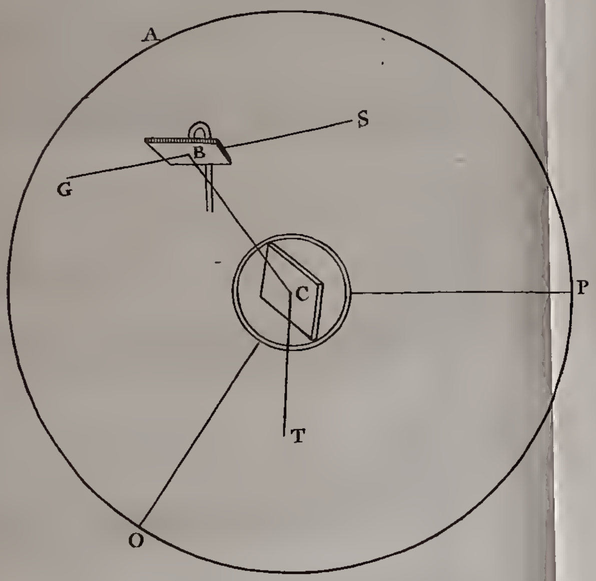


Fig. II.

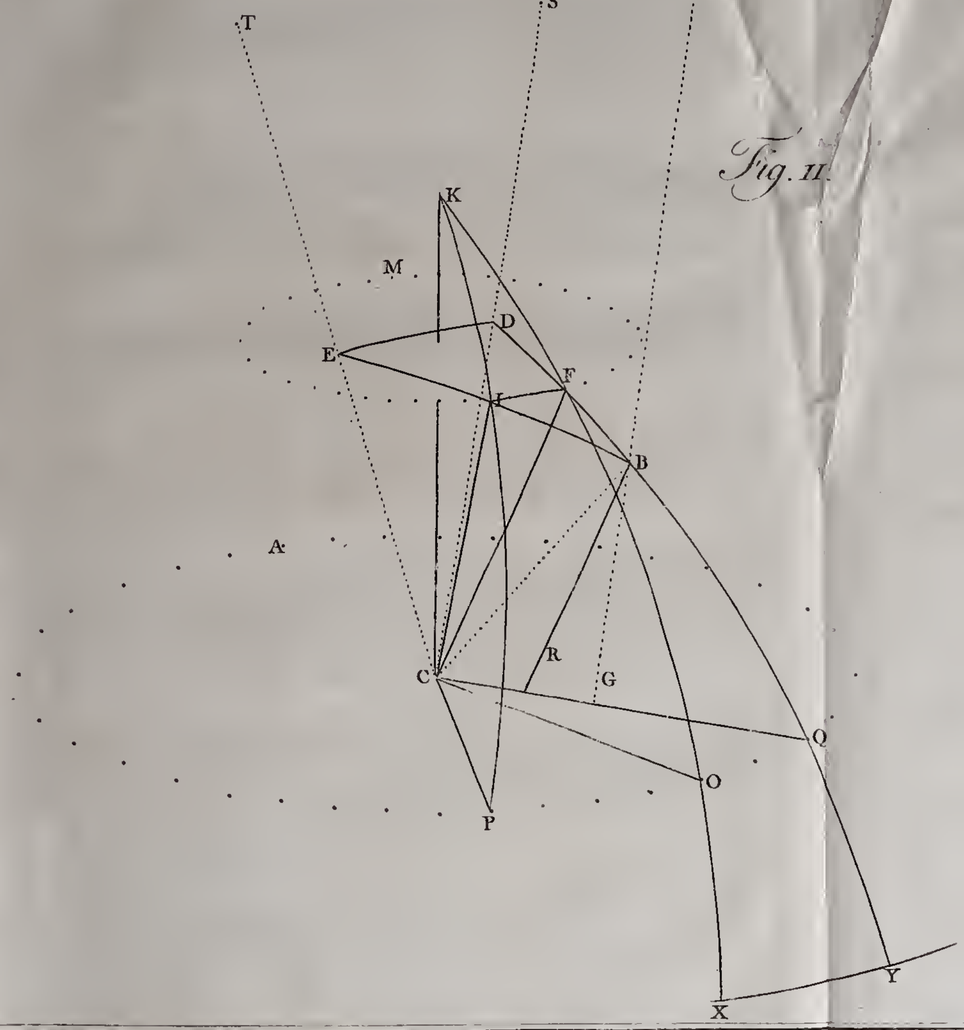


Fig. III.

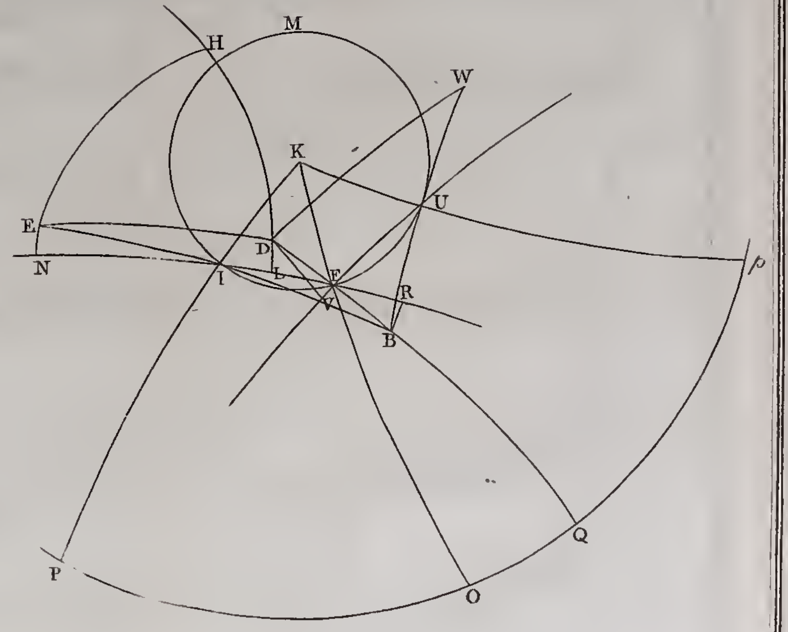


Fig. IV.

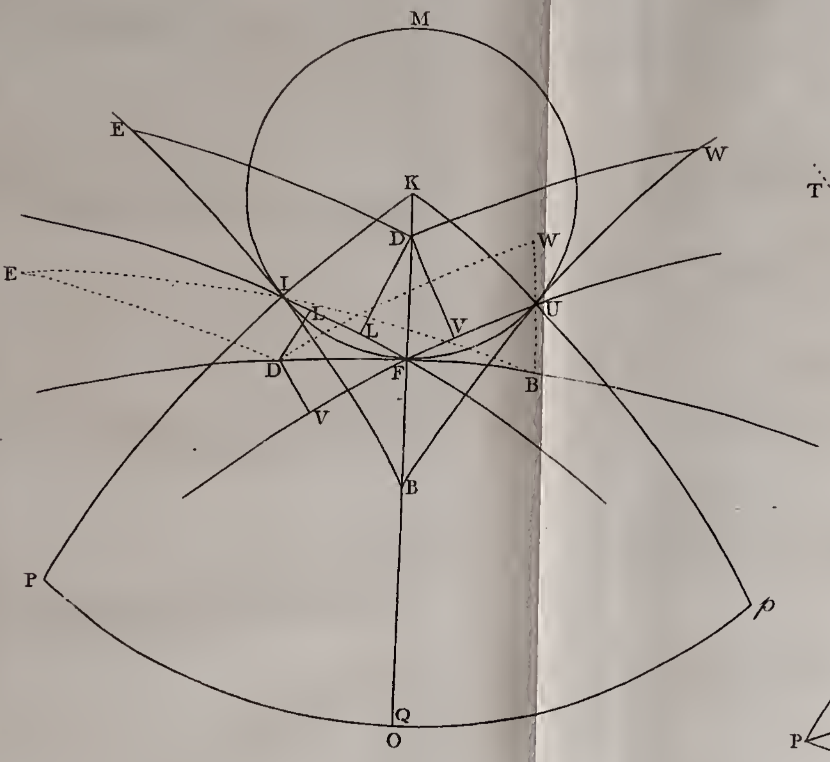


Fig. V.

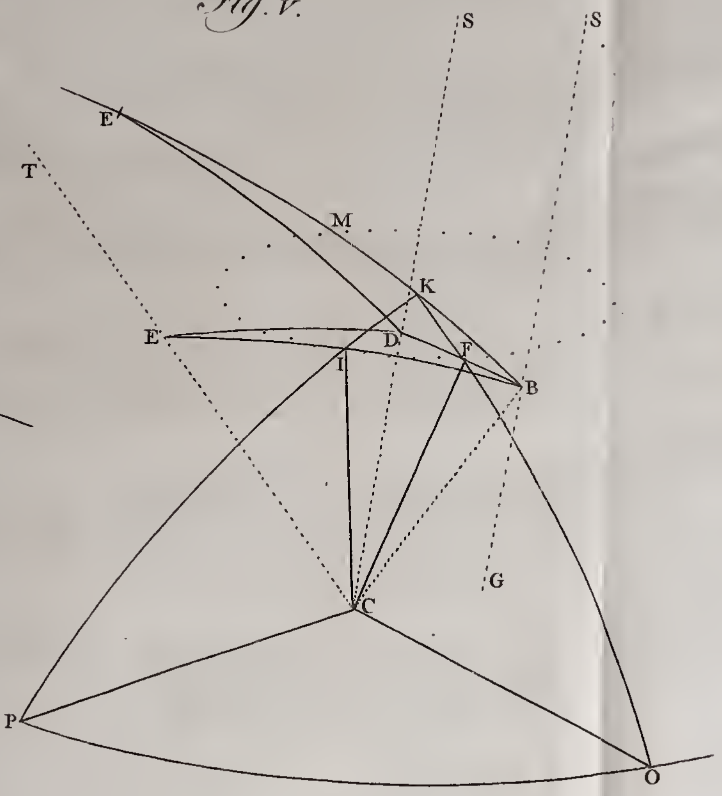
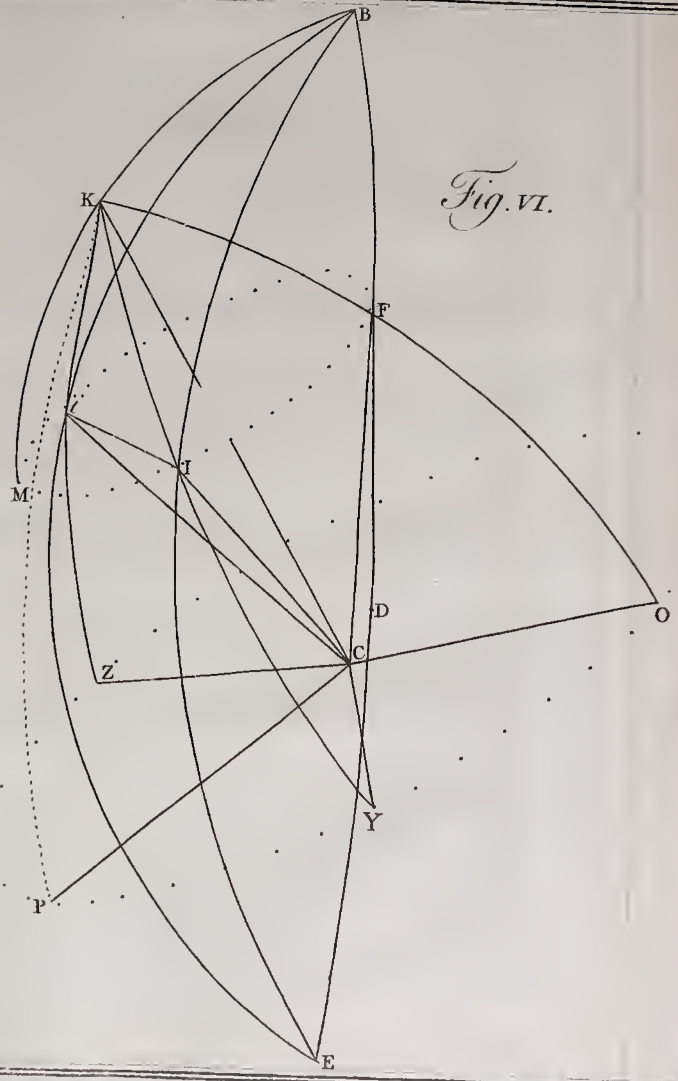


Fig. VI.



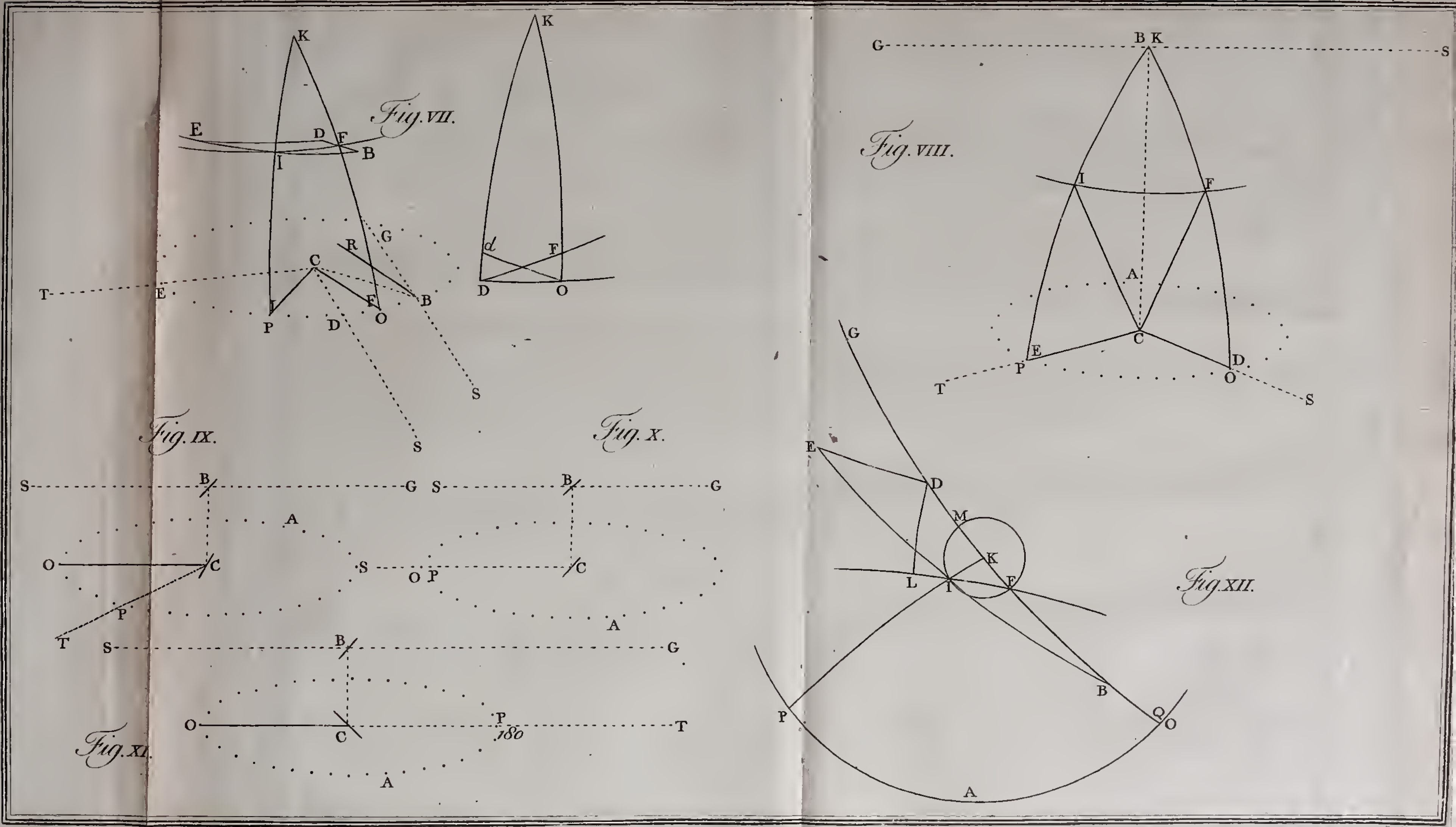


Fig. VII. Fig. VIII. Fig. IX. Fig. X. Fig. XI. Fig. XII.

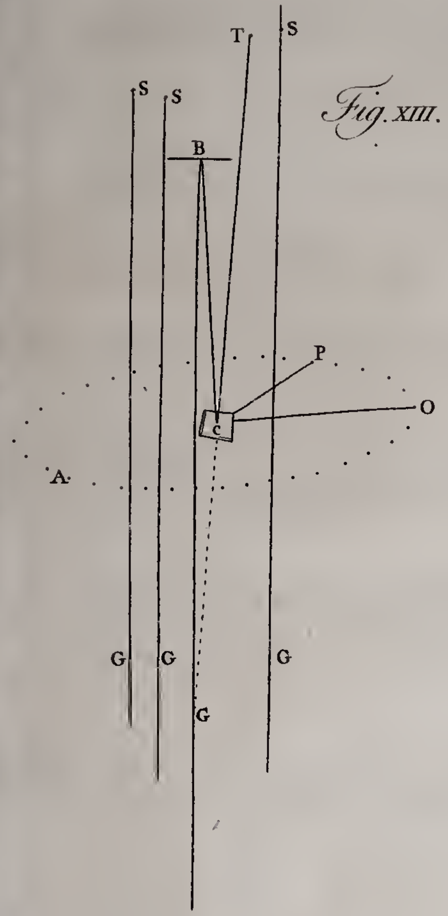


Fig. XIII.

Fig. XIV.

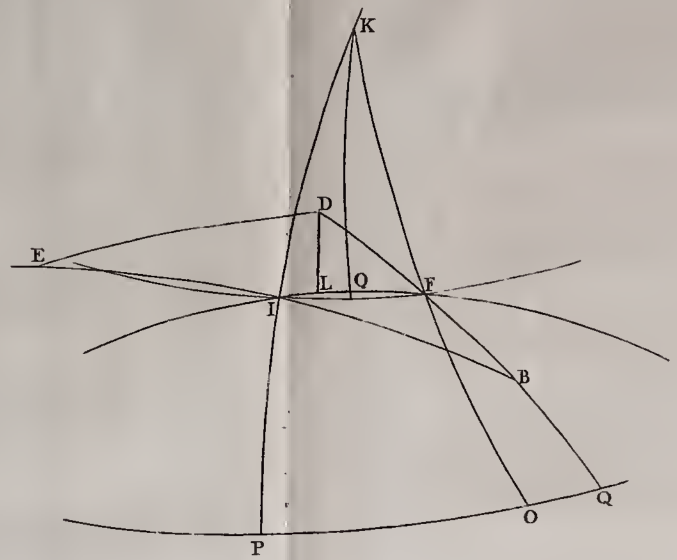
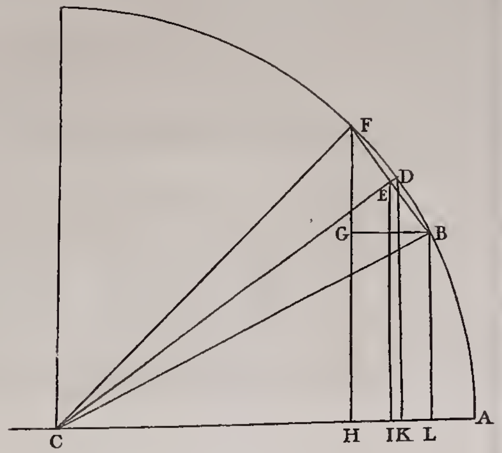


Fig. XV.



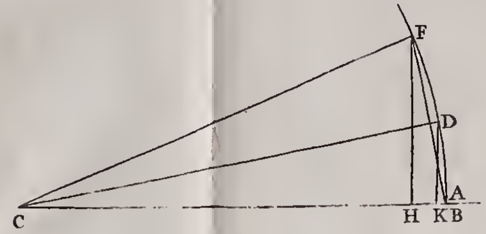


Fig. XVI.